On a constrained variational problem in the
vector-valued case

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Abstract

We prove some existence and regularity results for minimizers of a class of integral functionals, defined on vector-valued Sobolev functions \( u \) for which the volumes of certain level-sets \( \{ u = l_i \} \) are prescribed, with \( i = 1, \ldots, m \). More specifically, in the case of the energy density \( W(x, u, Du) = |Du|^2 + \beta F(u) \), we prove that minimizers exist and are locally Lipschitz, if the function \( F \) and \( \{ l_1, \ldots, l_m \} \) verify suitable hypotheses.

Résumé

On montre quelques résultats d'existence et régularité pour les minima d'une classe de fonctionnelles intégrales, définies sur les fonctions de Sobolev à valeurs vectorielles \( u \), telles que le volume de l'ensemble de niveau \( \{ u = l_i \} \) soit assigné, pour \( i = 1, \ldots, m \). Plus précisément, dans le cas où les fonctionnelles soient du type \( \mathcal{E}(u) = \int (|Du(x)|^2 + \beta F(u(x)))
\) dx, on prouve l'existence d'une fonction minimisante localement Lipschitzienne, des que \( F \) et \( \{ l_1, \ldots, l_m \} \) vérifient certaines hypothèses.

1 Introduction

One of the main tasks in the calculus of variations is to minimize functionals of the type

\[
\mathcal{E}(u) = \int_{\Omega} W(x, u, Du),
\]

among functions \( u \) verifying, for instance, some conditions at the boundary of the domain \( \Omega \). A huge variety of problems, arising from physics and material sciences, can be formulated in terms of scalar or vector-valued functions and of associated energies like (1). Here, our attention is focused on a class of free boundary and shape optimization-type problems, related to heat-flow through partially insulating materials (see [2]) and, in a certain asymptotic sense, to models for systems of immiscible fluids (see [5]). More precisely, we can state the problem as follows: let \( \Omega \subset \mathbb{R}^n \) be open and bounded, and let \( l_1, \ldots, l_m \in \mathbb{R}^d \) be fixed, together with corresponding real numbers \( v_1, \ldots, v_m > 0 \), such that the following compatibility condition holds:

\[
\sum_{i=1}^{m} v_i < |\Omega|.
\]
The central idea is to solve a new problem \( (M_\lambda) \), where \( \lambda \) is the penalization coefficient. We seek a \( H^1 \)-continuous solution, which turns out to solve also the original problem \( (M) \), provided we consider the problem

\[
\min_{u \in \mathcal{K}} \mathcal{E}(u).
\]

Here we denote by \( \mathcal{K} \) the whole space \( W^{1,p}(\Omega) \) of functions \( u \) such that \( |L_i(u)| = v_i \) for all \( i = 1, \ldots, m \).

One can immediately observe that, thanks to (2), \( \mathcal{K} \) is nonempty for any \( p \in [1, \infty] \). Then, we consider the problem

\[
\min_{u \in \mathcal{K}} \mathcal{E}(u).
\]

Note that, at this stage, no extra boundary conditions are required: indeed, we only ask that the volume (but not the shape!) of each level-set \( L_i(u) \) equals the prescribed value \( v_i \). In this sense, \( (M) \) belongs to a wider class of free boundary-type problems, which has been investigated by several authors: we mainly refer to works by [3], [4] and, for the case of volume constraints on several level-sets, to [5]. In that paper, under suitable assumptions on \( W \) as well as on the prescribed levels \( v_i \), an existence result is obtained in the vector-valued case, with the help of a weak formulation \( (M^*) \) where the constraint is relaxed, in order to ensure existence of weak solutions (see Section 2 and Theorem 3.1). However, the need to assume that the levels \( \{l_i\} \) are extremal points of their own convex hull imposes a strong limitation to that result, since, for instance, one cannot prescribe more than 2 levels in the scalar-valued case. In the following papers [8] and [7], existence and Hölder-continuity of solutions is proved in the scalar case without any extra hypothesis on \( \{l_i\} \), and for a larger class of integrands \( W \) satisfying a so-called flatness property (saying that, roughly speaking, any weak solution to the Euler-Lagrange equation associated to the functional on a ball \( B \subset \Omega \) either is constant on \( B \) or its level-sets have zero Lebesgue measure on \( B \)). The central idea is to solve a new problem \( (M_\lambda) \), that is, to minimize a penalized functional over the whole space \( W^{1,p}(\Omega) \): this leads, at least for a quite relevant class of integrands, to a Hölder-continuous solution, which turns out to solve also the original problem \( (M) \), provided the penalization coefficient \( \lambda \) is large enough. Moreover, in the case of two prescribed scalar levels, and for \( W(x,u,\nabla u) = |\nabla u|^2 \), the local Lipschitz-continuity of the minimizers is proved, too.

Here we obtain existence and regularity results in the vector-valued case, essentially by generalizing the technique of [8]. After some definitions and basic results, that have been collected in Section 2, in the next Section 3 we focus our attention on the relationships between \( (M) \), \( (M^*) \) and \( (M_\lambda) \), looking for properties on the integrand \( W(x,u,\xi) \) that guarantee the equivalence of \( (M^*) \) and \( (M_\lambda) \), at least when \( \lambda \) is large enough. This is a crucial step, since in some cases (see Section 3 for a model one) we recover solutions to \( (M_\lambda) \) by combining the (somehow guaranteed) continuity of solutions to \( (M_\lambda) \) (or \( (M^*) \)) with an argument involving the properties of the solutions to the Euler-Lagrange system associated with the functional \( \mathcal{E}(u) \). By this argument, in the spirit of the flatness property of [7], we shall be able to conclude that there are no level-sets \( L_i(u) \) with Lebesgue measure exceeding its prescribed value \( v_i \), i.e., that \( u \) actually solves the initial problem \( (M) \). We single out a condition on \( W \), called stretching property (see Definition 3.1) and which is sufficient for the equivalence of \( (M^*) \) and \( (M_\lambda) \) (see Theorem 3.2), then we consider two groups of structure conditions on \( W \) that imply the stretching property and, at the same time, can be checked more easily (see also Remark 3.3). Finally, in Section 4 we apply the results of the previous section to the case \( W(x,u,\xi) = |\xi|^2 + \beta F(u) \), where \( \beta \geq 0 \) and \( F : \mathbb{R}^d \to \mathbb{R} \) is a convex, coercive function of class \( C^1 \) having non-vanishing gradient at each prescribed level \( l_i \). Here, we are able to prove existence and Hölder continuity of minimizers (Theorem 4.1) and, furthermore, a regularity result (Theorem 4.2) stating that, under suitable assumptions on \( F \) and
the \{l_i\}'s, any solution is locally Lipschitz continuous near each level-set corresponding to an extremal level \(l_i\) (i.e., extremal with respect to the convex hull of \(\{l_1, \ldots, l_m\}\)). It is, of course, worth to point out that the extremality condition on the prescribed levels, originally considered in [5] for the existence of solutions, is recovered here as essential tool for proving Lipschitz-continuity. We conclude with Corollary 4.2, where the local Lipschitz-continuity of solutions is obtained in the specific case \(F(u) = f(|u - L|)\), with \(f : [0, \infty) \to \mathbb{R}\) a \(C^1\), convex and increasing function such that \(f'(0) = 0\), and with \(L \in \mathbb{R}^d\) such that all \(\{l_i\}\)'s are extremal points of the convex hull of \(\{L, l_1, \ldots, l_m\}\).

2 Definitions and preliminary results

Throughout the paper, \(\Omega\) will denote an open, bounded and connected subset of \(\mathbb{R}^n\), with Lipschitz boundary. If \(A \subset \mathbb{R}^n\) we write \(|A|\) for its \(n\)-dimensional Lebesgue measure. Given a positive integer \(N\), \(y \in \mathbb{R}^N\) and \(r > 0\), we denote by \(|y|\) the Euclidean norm of \(y\), and by \(B_r(y)\) the Euclidean open ball of radius \(r\) and center \(y\) in \(\mathbb{R}^N\). Given a positive integer \(d\), by \(\mathbb{R}^{d \times N}\) we denote the space of \((d \times N)\)-matrices \(\xi\) with real entries, whose Euclidean norm is again denoted by \(|\xi|\) (in this case we view \(\xi\) as a vector of \(d \cdot N\) components). The transposed of a matrix \(\xi\) will be denoted by \(\xi^t\), while \(I_d\) will denote the identity matrix in \(\mathbb{R}^{d \times d}\). Let \(M_1, M_2 \in \mathbb{R}^{d \times d}\) be two symmetric matrices, then we shall write \(M_1 \leq M_2\) whenever \(M_2 - M_1\) is nonnegative definite, that is, if its eigenvalues are nonnegative. We shall also consider the tensor product of two vectors \(a, b \in \mathbb{R}^d\), denoted by \(a \otimes b\) and represented by the matrix \((a_i b_j)_{i,j=1,\ldots,d}\). For a measurable function \(u : \Omega \to \mathbb{R}^d\), we denote by \(|u|^p\) its \(L^p\) norm, with \(p \in [1, +\infty]\). We also consider vector-valued Sobolev spaces \(W^{1,p}(\Omega, \mathbb{R}^d)\) and, in the case \(d = 1\), we adopt the shorter notation \(W^{1,p}(\Omega)\). We recall here a useful version of the Poincaré inequality: take \(p \geq 1\), \(\alpha > 0\) and \(l \in \mathbb{R}\), then there exists a constant \(C = C(\alpha, p, \Omega, n) > 0\) such that

\[
\|w - l\|_p \leq C \|\nabla w\|_p,
\]

for all \(w \in W^{1,p}(\Omega)\) verifying \(|\{w = l\}| \geq \alpha\) (from now on, we shall often write \(\{w = l\}\) instead of \(\{x \in \Omega : w(x) = l\}\)). We will sometimes need the localized form of the energy \(\mathcal{E}(u)\): more precisely, given \(A \subset \Omega\) measurable, we define

\[
\mathcal{E}(u, A) := \int_A W(x, u, Du).
\]

Moreover, we shall frequently denote by \(C\) a generic, positive constant, depending only on the data of the problem and whose value can change from one line to another.

We will assume the following properties on the integrand \(W\) (see [6]):

**Hypothesis 2.1** \(W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times n} \to \mathbb{R}\) is a quasiconvex Carathéodory function satisfying

(i) for some \(1 < p < \infty\), there exists a constant \(C > 0\) such that

\[
\frac{1}{C} |\xi|^p - C \leq W(x, u, \xi) \leq C(1 + |\xi|^p)
\]

for all \((x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times n}\);
(ii) $W(x, \cdot, \cdot)$ is locally Lipschitz for all $x \in \Omega$, and satisfies

$$|W(x, u, \xi) - W(x, v, \xi)| \leq C(1 + |\xi|^p)|u - v|.$$  

**Remark 2.2** If $W$ satisfies Hypothesis 2.1 above, then the sequential weak lower semicontinuity in $W^{1,p}(\Omega, \mathbb{R}^d)$ of the functional $\mathcal{E}(u)$ follows from [1]. Moreover, the functional $\mathcal{E}(u)$ is coercive, that is, $\mathcal{E}(u) \geq \frac{1}{\ell} \|u\|_{W^{1,p}}^p - C$ as soon as the Lebesgue measure of the level-set $L_i(u)$ is greater than some positive constant (this latter assertion can be proved by combining Hypothesis 2.1 (i) with the Poincaré inequality (3)).

Now, the fact that the lower semicontinuity of the energy coupled with growth conditions is not enough to prove existence of solutions to the initial problem (M) via direct methods is easily understood by noting that the volume constraints $|L_i(u)| = v_i$ are not necessarily attained by the weak limit $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ of a (minimizing) sequence $(u_h)_h$. Indeed, one has only the following estimate for all $i = 1, \ldots, m$ (see [5]):

$$\limsup_h |L_i(u_h)| \leq |L_i(u)|. \tag{4}$$

To overcome the fact that $\mathcal{K}$ is not closed with respect to weak convergence in $W^{1,p}(\Omega, \mathbb{R}^d)$, one can relax the initial problem (M) by taking into account the information given by (4). This technique has been used by Ambrosio et al. in [5], where the following relaxed problem is considered:

$$\min_{u \in \mathcal{K}^*} \mathcal{E}(u), \tag{M^*}$$

where

$$\mathcal{K}^* = \{ u \in W^{1,p}(\Omega, \mathbb{R}^d) : |L_i(u)| \geq v_i \ \forall \ i = 1, \ldots, m \}. \tag{5}$$

Hence, we still have a constrained problem, but now the constraint is closed with respect to weak convergence, and this will eventually lead to existence of weak (relaxed) solutions.

A different approach has been adopted in [8] and [7] to tackle the scalar-valued case. It consists of minimizing a new energy $\mathcal{E}_\lambda(u)$ over the entire space $W^{1,p}(\Omega)$, defined as the sum of $\mathcal{E}(u)$ with an extra term $\mathcal{P}_\lambda(u)$ that adds a penalization when the volume of some level-set $L_i(u)$ is strictly less than its prescribed value $v_i$:

$$\min_{u \in W^{1,p}(\Omega)} \mathcal{E}_\lambda(u), \tag{M_\lambda}$$

where

$$\mathcal{E}_\lambda(u) := \mathcal{E}(u) + \mathcal{P}_\lambda(u) \quad \text{and} \quad \mathcal{P}_\lambda(u) := \lambda \sum_{i=1}^m (v_i - |L_i(u)|)^+,$$

with $\lambda > 0$ and $z^+$ denoting the positive part of $z \in \mathbb{R}$. This clearly represents a weaker form of problem (M$^*$) itself: indeed, if the volume of $L_i(u)$ is greater than or equal to the prescribed value $v_i$ for all $i$, then $\mathcal{P}_\lambda(u) = 0$, hence $\mathcal{E}_\lambda(u) = \mathcal{E}(u)$ and we come back to (M$^*$) (see also Remark 2.4).

It is convenient to define some quantities related to the problems (M), (M$^*$) and (M$\lambda$):

$$v_{\min} := \min_{1 \leq i \leq m} v_i, \tag{6}$$

$$\mu := \inf_{u \in \mathcal{X}} \mathcal{E}(u), \quad \mu^* := \inf_{u \in \mathcal{X}^*} \mathcal{E}(u), \quad \mu_\lambda := \inf_{u \in W^{1,p}(\Omega, \mathbb{R}^d)} \mathcal{E}_\lambda(u). \tag{7}$$
Proposition 2.3 Suppose that Hypothesis 2.1 holds, then one has
\[ \mu_\lambda \leq \mu^* \leq \mu \] \hfill (8)
and, for each solution \( u \) to (\( M_\lambda \)),
\[ v_{\text{min}} - \frac{\mu^*}{\lambda} \leq |L_i(u)|, \quad i = 1, \ldots, m. \] \hfill (9)
Moreover, both problems (\( M^* \)) and (\( M_\lambda \)) admit at least a solution if \( \lambda \) is large enough.

Proof. One can deduce (8) and (9) as follows: (8) is a straightforward consequence of the definition and the fact that \( K \subset K^* \) and \( P_\lambda(u) = 0 \) whenever \( u \in K^* \), while to get (9) one can simply use the minimality of \( u \) and (8), to obtain
\[ \lambda(v_i - |L_i(u)|) \leq P_\lambda(u) \leq \mathcal{E}_\lambda(u) = \mu_\lambda \leq \mu^*, \]
which gives (9) at once. As for the existence of solutions, both functionals \( \mathcal{E}(u) \) and \( \mathcal{E}_\lambda(u) \) are lower semicontinuous, thanks to Hypothesis 2.1. The coercivity of \( \mathcal{E}(u) \) over \( K^* \) follows from Remark 2.2 (indeed, \( u \in K^* \) verifies \( |L_i(u)| \geq v_{\text{min}} \)), while that of \( \mathcal{E}_\lambda(u) \) over \( W^{1,p}(\Omega, \mathbb{R}^d) \) can be seen by restricting the minimization to functions \( u \) such that \( \mathcal{E}_\lambda(u) \leq \mu_\lambda + 1 \leq \mu^* + 1 \). For these functions, one obtains \( |L_i(u)| \geq v_{\text{min}} - \frac{\mu^* + 1}{\lambda} \), thus if \( \lambda \geq \frac{2\mu^* + 2}{v_{\text{min}}} \) then \( |L_i(u)| \geq v_{\text{min}}/2 \) and therefore by Remark 2.2 we deduce the coercivity of \( \mathcal{E}_\lambda(u) \). Then, the existence of solutions to (\( M^* \)) and (\( M_\lambda \)) follows from the application of the direct method of calculus of variations.

Remark 2.4 If there exists a solution \( u_\lambda \) to (\( M_\lambda \)) for which \( P_\lambda(u_\lambda) = 0 \), then we conclude that \( u_\lambda \) solves (\( M^* \)), too; moreover, we get \( \mu_\lambda = \mu^* \), hence any other solution to (\( M^* \)) is also a solution to (\( M_\lambda \)). On the other hand, there is no a-priori guarantee that all solutions to (\( M_\lambda \)) are automatically solutions to (\( M^* \)).

3 Relations between (\( M \)), (\( M^* \)) and (\( M_\lambda \))

By Proposition 2.3 we know that (\( M^* \)) and (\( M_\lambda \)) admit solutions. The problem is now to prove that, under some extra conditions, such solutions solve also (\( M \)). A first possible strategy is represented by Theorem 3.2 of [5], that we quote here:

Theorem 3.1 Let \( u \) be a solution to (\( M^* \)) and suppose that \( W \) depends only on the variable \( \xi \) and that

(a) \( W \) is differentiable and \( \sum_{i=1}^{d} \sum_{k=1}^{n} \frac{\partial W}{\partial \xi_{ik}} \leq C(1+|\xi|^{p-1}) \) for some \( C > 0 \) and all \( \xi \in \mathbb{R}^{d \times n} \);

(b) \( \sum_{i,j=1}^{d} \sum_{k=1}^{n} \frac{\partial W}{\partial \xi_{ik}} \xi_{ij} \nu_i \nu_j > 0 \) whenever \( \xi^t \nu \neq 0, \xi \in \mathbb{R}^{d \times n}, \nu \in S^{d-1} \);

(c) \( l_1, \ldots, l_m \) are extremal points of their convex hull.
Then $u$ is a solution to $(M)$. We remark that this result does not require any continuity property of the solutions to $(M^*)$, but the extremality condition (c) on the prescribed levels represents a strong constraint (for instance, in the scalar-valued case one can prescribe only two levels).

Another strategy has been used in [7] for the scalar-valued case. The idea is, first of all, to consider the penalized problem $(M_\lambda)$ and prove that, for $\lambda$ large enough, any solution $u$ to $(M_\lambda)$ actually solves $(M^*)$: this step is quite crucial and will be object of analysis in the present section. Then, one tries to prove that $(M_\lambda)$ admits continuous solutions, which is in general much easier than for the constrained problem $(M^*)$, and finally one tries to exclude that $|L_i(u)| > v_i$ for some $i$, thus concluding that $|L_i(u)| = v_i$ for all $i$, i.e., that $u$ solves $(M)$. This last step will require special conditions on the integrand $W$, such as the so-called flatness property (see [7]) saying that, roughly speaking, any weak solution $u$ to the Euler-Lagrange equation (system) associated to $\mathcal{E}(\cdot)$ in a ball $B \subset \Omega$ either is constant on $B$ or has all level-sets of zero Lebesgue measure inside $B$. See, however, the model case discussed in Section 4 (and especially Theorem 4.1 and its proof).

In the rest of the section we shall be mainly concerned with the discussion of the equivalence of $(M^*)$ and $(M_\lambda)$ for $\lambda$ large. Such equivalence will follow as soon as we guarantee that the energy $\mathcal{E}(u)$ increases in a controlled way after a suitable stretching of $u$ near some chosen level. To state this in a more precise way, we introduce some notation. We fix $0 < \delta < \frac{1}{2}$ and, for a given $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, we define

$$u_\delta(x) = \begin{cases} 
 u(x) & \text{if } u(x) \notin B_1 \\
 0 & \text{if } u(x) \in \overline{B}_\delta \\
 \frac{1}{1 - \delta} \left(1 - \frac{\delta}{|u(x)|}\right) u(x) & \text{if } u(x) \in B_1 \setminus \overline{B}_\delta.
\end{cases}$$

(10)

In other words, $u_\delta$ is obtained by stretching $u$ at 0, and can be written as $T_\delta(u)$, with $T_\delta : \mathbb{R}^d \rightarrow \mathbb{R}^d$ identically 0 on $B_\delta$, coinciding with the identity out of $B_1$, and “radially affine” on $B_1 \setminus B_\delta$. Note that $u_\delta$ still belongs to $W^{1,p}(\Omega, \mathbb{R}^d)$, since $T_\delta$ is a Lipschitz map. It is also worth calculating the weak gradient of $u_\delta$: \n
$$Du_\delta(x) = DT_\delta(u) \cdot Du(x) = \begin{cases} 
 Du(x) & \text{if } u(x) \notin B_1 \\
 0 & \text{if } u(x) \in \overline{B}_\delta \\
 \frac{1}{1 - \delta} M_\delta(u) \cdot Du(x) & \text{if } u(x) \in B_1 \setminus \overline{B}_\delta,
\end{cases}$$

(11)

where

$$M_\delta(u) := \frac{\delta}{|u|} \left(\frac{u \otimes u}{|u|^2}\right) + \left(1 - \frac{\delta}{|u|}\right) I_d \in \mathbb{R}^{d \times d}.\quad(12)$$

In a similar way, we can define $u_{\delta l}$ as a stretching of $u$ around a generic level $l \in \mathbb{R}^d$, by setting $u_{\delta l} = l + T_\delta(u - l)$. Then, we are led to the following definition of stretching property.

**Definition 3.1 (Stretching property)** Given $p, \Omega$ as before, we say that $W$ satisfies the stretching property if there exists a constant $C = C(p, \Omega) > 0$, such that, for all $\varepsilon < 1$, $t \in [0, 1]$, $\nu \in \mathbb{S}^{d-1}$, and $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times n}$, one has

$$W(x, u, (1 + \varepsilon)M^*_\varepsilon \xi) - W(x, u, \xi) \leq C(1 + |\xi|^p) \varepsilon,$$

(13)

where $M^*_\varepsilon := t(\nu \otimes \nu) + (1 - t)I_d$. 


The stretching property (13) gives a sufficient condition for the equivalence of the two problems (Mλ) and (M∗), as showed by the following result:

**Theorem 3.2** Suppose that W verifies Hypothesis 2.1 and (13). Then, there exists a constant C > 0 such that, for all λ > C, any solution u to (Mλ) is also a solution to (M∗).

**Proof.** By (9) we infer that, for λ > 2vmin/κ, any minimizer u of (Mλ) verifies |Lj(u)| ≥ vmin/2 for all j = 1, . . . , m. Moreover, the coercivity of Eλ(·) (see Remark 2.2) tells us that

\[ \|u\|_{W^{1,p}} \leq C \]  \hspace{1cm} (14)

for some C. We only need to prove that if the strict inequality |Lj(u)| < vj is verified for some i, then λ must be bounded by a constant independent on u. Again, we suppose j = 0 and that min j̸=i |lj| > 1 (true up to scaling). We then consider the function uδ defined in (10), with 0 < δ < 1/p so small that

\[ |Lj(u)| < |Lj(uδ)| = |\{|u(x)| \leq \delta\}| \leq v_j. \]

By the fact that Eλ(u) ≤ Eλ(uδ), and setting Dδ = {δ < |u| < 1} and ΔE = E(uδ, Dδ) − E(u, Dδ), we obtain

\[ E(u, \{|u(x)| \leq \delta\}) + \lambda \{0 < |u(x)| \leq \delta\} \leq \Delta E. \]  \hspace{1cm} (15)

Now, we claim that

\[ \Delta E \leq C\delta. \]  \hspace{1cm} (16)

Indeed, thanks to Hypothesis 2.1 (ii), (14) and observing that |Duδ| ≤ |Du|/(1 − δ), we have

\[ \Delta E \leq \int_{D_\delta} \left( W(x, u, Du_\delta) - W(x, u, Du) \right) \geq C\delta \int_{D_\delta} \left( 1 + \frac{|Du_\delta|}{1 - \delta} \right) \leq \int_{D_\delta} \left( W(x, u, Du_\delta) - W(x, u, Du) \right) + C\delta, \]  \hspace{1cm} (17)

Finally, by (17), (11), and thanks to (13) applied with ε = 1/(1−δ), we easily obtain (16).

The growth condition (i) of Hypothesis 2.1 and (15) combined with (16) imply

\[ \int_{\{0 < |u(x)| \leq \delta\}} (|Du(x)|^p + \lambda) \, dx \leq C(\delta + \{|0 < |u(x)| \leq \delta\}). \]  \hspace{1cm} (18)

Now, if δ < |\{0 < |u(x)| \leq \delta\}| then we get immediately the uniform upper bound λ ≤ 2C, which would conclude the proof, while if |\{0 < |u(x)| \leq \delta\}| ≤ δ then (18) becomes

\[ \int_{\{0 < |u(x)| \leq \delta\}} (|Du(x)|^p + \lambda) \, dx \leq C\delta, \]  \hspace{1cm} (19)

hence we can proceed as follows. Since for a, b ≥ 0 the inequality \( a^{1-\frac{1}{p}}b \leq a + b^p \) holds, from (19) we infer

\[ \lambda^{1-\frac{1}{p}} \int_{\{0 < |u(x)| \leq \delta\}} |Du(x)| \, dx \leq C\delta. \]  \hspace{1cm} (20)
Then, we define \( w(x) = f(|u(x)|) \), with
\[
f(t) = \begin{cases} 
  t & \text{if } 0 \leq t \leq \delta \\
  \delta & \text{if } t > \delta,
\end{cases}
\]
and by using Fubini's theorem and Poincaré inequality (3), together with \( |Du| \leq |Du| \), we obtain
\[
\int_0^\delta \{ w \geq t \} \, dt = \int_{\Omega} w \leq C \int_{\Omega} |\nabla w| \leq C \int_{\{0 < |u(x)| \leq \delta\}} |Du|.
\] (21)
On the other hand, the inclusion
\[
\bigcup_{j \neq i} L_j(u) \subset \{ w \geq t \},
\]
holds for all \( t < \delta \), therefore (21) and (20) give
\[
\frac{\nu_{\min}}{2} \lambda^{\frac{1}{p'}} \leq C \delta,
\]
We finally deduce that \( \lambda < \left( \frac{2C}{\nu_{\min}} \right)^{\frac{1}{p'}} \), and this concludes the proof. \( \square \)

In what follows, we consider two pairs of structure conditions on \( W \), implying the stretching property (13) when coupled with Hypothesis 2.1 (as shown in Proposition 3.5). The first pair \((A1) - (A2)\) generalizes, respectively, the hypotheses (a) and (b) of Theorem 3.1 (see Proposition 3.4), while the second pair \((B1) - (B2)\) provides alternative conditions that seem to identify a slightly different class of integrands satisfying (13).

(A1) \( W \) is differentiable and satisfies
\[
\sum_{i=1}^d \sum_{k=1}^n \left| \frac{\partial W}{\partial \xi_{ik}}(x, u, \xi) \right| \leq C(1 + |\xi|^{p-1})
\]
for some \( C > 0 \) and all \((x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times n}\);

(A2) \( W(x, u, \xi) \leq W(x, u, M\xi) \), for every \((x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times n}\) and every symmetric matrix \( M \in \mathbb{R}^{d \times d}\) verifying \( M \geq I_d\);

(B1) \( W(x, u, \cdot) \) is convex and takes its minimum at \( \xi = 0 \), for all \((x, u) \in \Omega \times \mathbb{R}^d\);

(B2) \( W(x, u, \nu \otimes \nu \xi) \leq W(x, u, \xi) \), for all \((x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times n}\) and all \( \nu \in \mathbb{R}^d \) such that \( |\nu| = 1 \).

Remark 3.2 (A2) implies (B2). Indeed, the rank-one matrix \( \nu \otimes \nu \) can be approximated by the sequence of invertible matrices \( N_\varepsilon = \varepsilon I_d + (1 - \varepsilon) \nu \otimes \nu \), as \( \varepsilon \to 0^+ \), hence (B2) will follow once we prove
\[
W(x, u, \xi) \leq W(x, u, N_\varepsilon^{-1} \xi).
\] (22)
Now, since \( N_\varepsilon \leq I_d \) we deduce that \( N_\varepsilon^{-1} \geq I_d \), and therefore by (A2) we get (22), as wanted.
Remark 3.3 The integrand $W(x,u,\xi) = |\xi|^p + F(x,u)$, with $F$ bounded, measurable in $x$ and uniformly Lipschitz in $u$, satisfies Hypothesis 2.1, (A1) and (A2), for any $p > 1$. Indeed, Hypothesis 2.1 and (A1) are quite immediate to check, while for (A2) one can observe that, given a symmetric matrix $M \geq I_d$, there exists an orthogonal matrix $U$ such that $M = U^{-1}AU$, where $A$ is a diagonal matrix with eigenvalues greater than 1. Since the Hilbert (Euclidean) norm $|.|$ is invariant under orthogonal transformations, setting $\xi' = U\xi$ one gets

$$|M\xi| = |U^{-1}AU\xi| = |A\xi'| = |\xi|,$$

hence $|\xi|^p + F(x,u) \leq |M\xi|^p + F(x,u)$, as wanted.

In the next proposition we show that property (A2) can be derived starting from hypothesis (b) of Theorem 3.1.

Proposition 3.4 Let $W$ satisfy Hypothesis 2.1, be differentiable in $\xi$ and verify condition (b) of Theorem 3.1. Then it verifies (A2).

Proof. As before, we simplify the notation by supposing that $W$ depends only on $\xi$. First, we prove that (b) implies the following relation:

$$W(\eta_t) - W(\xi) \geq \nabla W(\xi) \cdot (\eta_t - \xi) = t \sum_{i,j=1}^{d} \sum_{k=1}^{n} \frac{\partial W}{\partial \xi_{ik}}(\xi)\xi_{ik} \nu_i \nu_j > 0,$$

thus obtaining a strict inequality even stronger than (23). On the other hand, if $\xi'\nu = 0$ then we may approximate $\xi'$ by means of matrices $\xi_{\varepsilon}$, for which $\xi'_{\varepsilon}\nu \neq 0$, and finally take the limit as $\varepsilon$ goes to 0 and conclude by means of the continuity of $W$.

Now, we claim that (23) implies (A2). To see this, we fix a spectral basis $\{\nu^1, \ldots, \nu^d\}$ for $N = M - I_d$, so that we have the following representation:

$$N = \sum_{h=1}^{d} t_h \nu^h \otimes \nu^h,$$

where $t_h \geq 0$ is the eigenvalue relative to $\nu^h$. It is not difficult to show that

$$I_d + \sum_{h=1}^{k+1} t_h \nu^h \otimes \nu^h = (I_d + t_{k+1} \nu^{k+1} \otimes \nu^{k+1}) \cdot (I_d + \sum_{h=1}^{k} t_h \nu^h \otimes \nu^h),$$

for all $k = 1, \ldots, d-1$, hence by inductively applying (23) one obtains

$$W(\xi) \leq W((I_d + N)\xi) = W(M\xi),$$

as wanted. $\square$

Finally, we see in the following proposition that the two groups of structure conditions considered above imply the stretching property (13).
Proposition 3.5 Suppose that $W$ satisfies Hypothesis 2.1 and that, alternatively, either (A1) – (A2) or (B1) – (B2) are verified. Then $W$ satisfies (13).

Proof. From now on, to let the notation be more readable, we shall write $W(\xi)$ instead of $W(x, u, \xi)$, since $x$ and $u$ can be thought as fixed. We choose $\varepsilon, t \in (0, 1)$ and $\nu \in S^{d-1}$, then we want to prove (13), i.e.,

$$W((1 + \varepsilon)M'\nu t \xi) - W(\xi) \leq C(1 + |\xi|^p) \varepsilon$$

(24)

for all $\xi \in \mathbb{R}^{d \times n}$. We split the proof into two parts.

Part I. First, we suppose (A1) and (A2) to hold, then observe that $M'\nu$ is invertible and $\leq I_d$ (compare with Remark 3.2), hence we get from (A2)

$$W((1 + \varepsilon)M'\nu \xi) \leq W((1 + \varepsilon)\xi)$$

(25)

On the other hand, (A1) implies

$$W((1 + \varepsilon)\xi) - W(\xi) \leq \sum_{i=1}^{d} \sum_{k=1}^{n} \left| \frac{\partial W}{\partial \xi_k}(\xi') \right| \varepsilon |\xi|$$

$$\leq C \varepsilon |\xi|(1 + |\xi'|^{p-1}) \leq C \varepsilon |\xi|(1 + 2^{p-1}|\xi|^{p-1})$$

(26)

where $\xi'$ belongs to the segment joining $\xi$ and $(1 + \varepsilon)\xi$, hence (24) follows immediately from (25) and (26).

Part II. Now, we suppose that (B1) and (B2) are verified and, by applying (B1) and then (B2), we get (25) as before (recall that $M'\nu$ is a convex combination of $\nu \otimes \nu$ and $I_d$). Then, to obtain an estimate like (26) it is sufficient to observe that $(1 + \varepsilon)\xi = (1 - \varepsilon)\xi + \varepsilon(2\xi)$, therefore by (B1) we infer

$$W((1 + \varepsilon)\xi) \leq (1 - \varepsilon)W(\xi) + \varepsilon W(2\xi)$$

and, thanks to Hypothesis 2.1 (i),

$$W((1 + \varepsilon)\xi) - W(\xi) \leq \varepsilon(W(2\xi) - W(\xi)) \leq C \varepsilon (1 + |\xi|^p),$$

that is, the desired estimate. \qed

4 Existence and regularity of solutions in a special case

Let us consider, as an example, the case where $W(x, u, Du) = |Du|^2 + \beta F(u)$. Here the existence result follows:

Theorem 4.1 Suppose that $F: \mathbb{R}^d \rightarrow \mathbb{R}$ is $C^1$, convex, coercive, and such that $\nabla F(l_i) \neq 0$, for all $i = 1, \ldots, m$, and consider the energy density $W(x, u, \xi) = |\xi|^2 + \beta F(u)$, with $\beta \geq 0$. Then $(M)$ admits solutions, and every solution is bounded and locally Hölder continuous.
Proof. First, notice that the growth of $F(u)$ when $|u| \to \infty$ does not matter, since the minimization can be equivalently restricted to uniformly bounded functions, via projection onto a cube containing $\{l_1, \ldots, l_m\}$; indeed, if $Q$ is such a cube and $L : \mathbb{R}^d \to \mathbb{R}^d$ denotes the orthogonal projection on $Q$, then, arguing component by component, one can easily prove that $|D(L \circ u)| \leq |Du|$ almost everywhere on $\Omega$. The functional $\mathcal{E}(u) = \int_{\Omega} |Du|^2 + \beta F(u)$ is lower semicontinuous in the $W^{1,2}$ topology, and the hypotheses of Theorem 3.2 are fulfilled, at least when $|u|$ is bounded. Let $u$ be any (bounded) solution to $(M)$. If $A$ is large enough, then $u$ solves $(M^\ast)$ by Theorem 3.2. Now we prove that $u$ is continuous. Choose any ball $B_r$ in $\Omega$ and, there, replace $u$ by the harmonic function $v$ coinciding with $u$ along $\partial B_r$. Then $\mathcal{P}_\lambda$ increases at most by $\lambda |B_r|$, hence the minimality of $u$ yields
\[
\int_{B_r} |Du - Dv|^2 = \int_{B_r} |Du|^2 - |Dv|^2 \leq \lambda |B_r| + \beta \int_{B_r} F(v) - F(u).
\]
Since $|u| \leq C$, also $|v| \leq C$ and we see that the right hand side is $O(v^\alpha)$. Hence $u$ is locally $C^1$ for every $\gamma \in (0, 1)$. We are now ready to show that $u$ actually solves (M). Suppose that $\beta > 0$ (the case $\beta = 0$ is even simpler) and $|L_i(u)| > \alpha_i$ for some $i$, and let $A \subset \Omega$ be the open set where $|u - l_i| < \delta$ for some small $\delta > 0$. We claim that $u$ solves in $A$ the Euler system $2\Delta u = \beta \nabla F(u)$. Indeed, for every small ball $B \subset A$ such that $|B| < |L_i(u)| - \alpha_i$, the function $u_x = u + \varepsilon \eta$, with $\eta \in C^\infty_0(B)$ satisfies $\mathcal{P}_\lambda(u_x) = \mathcal{P}_\lambda(u)$ for small $\varepsilon$ such that $\varepsilon |\eta| < \delta$, hence $u$ is a local minimizer of $\mathcal{E}(\cdot, B)$ and the Euler system holds in $B$. Being a local property, it holds in the whole $A$. But now we get a contradiction, since a.e. on the level set $L_i(u)$ we have $\Delta u = 0$ (note that $u \in W^{2,2}_{loc}(A)$), whereas $\nabla F(l_i) \neq 0$ by assumption and this concludes the proof.

We conclude our analysis with a regularity result (Theorem 4.2), that holds under extra hypotheses on $l_1, \ldots, l_m$ and $F(u)$. Its proof will need the following lemma:

Lemma 4.1 Assume $u, v \in H^1(B_r(x_0))$, with $u, v \geq 0$, $v$ superharmonic and $u - v \in H^1_0(B_r(x_0))$. Then
\[
\left( \frac{1}{r} \int_{\partial B_r} u \right)^2 | \{u = 0\} \cap B_r | \leq C \int_{B_r} |\nabla (u - v)|^2,
\]
where $C$ depends only on the dimension.

For simplicity, we shall prove the lemma assuming that $u > 0$ in $B_r/2(x_0)$, since we shall need it only in this case. However, reasoning as in [4] (from which we take the main idea) one can adapt the proof to handle the general case.

Proof. By scaling, we may assume that $B_r(x_0) = B_1$ and $u > 0$ in $B_1/2$. For a.e. $\xi \in \partial B_1$, the restrictions of $u, v$ to the segment through $\xi$ and the origin are absolutely continuous and $u(\xi) = v(\xi)$. For such $\xi$, let $r_\xi$ denote the smallest value of those $r \in [1/2, 1]$ such that $u(r\xi) = 0$ if this set is non empty, $r_\xi = 1$ otherwise. Then we find
\[
v(r_\xi \xi) = v(r_\xi \xi) - u(r_\xi \xi) = \int_{r_\xi}^1 \frac{d}{dr}(u - v)(r\xi) dr \leq \sqrt{1 - r_\xi} \left( \int_{r_\xi}^1 |\nabla (u - v)(r\xi)|^2 dr \right)^{1/2}.
\]
If $h$ is the harmonic function in $B_1$ with boundary value $u$, we have $v(r_\xi \xi) \geq h(r_\xi \xi)$ since $v$ is superharmonic and, using the Poisson formula for $h$, we find
\[
v(r_\xi \xi) \geq h(r_\xi \xi) \geq c_n (1 - r_\xi) \int_{\partial B_1} u, \quad c_n > 0.
\]
If \( r_\xi < 1 \), we find using the last two inequalities

\[
(1 - r_\xi) \left( \int_{\partial B_r} u \right)^2 \leq C_n \int_{r_\xi}^1 |\nabla (u - v)(r\xi)|^2 \, dr,
\]

which is also valid if \( r_\xi = 1 \). By the definition of \( r_\xi \), we have

\[
(1 - r_\xi) \geq \int_{r_\xi}^1 \chi_{\{u=0\}}(r\xi) \, dr = \int_{1/2}^1 \chi_{\{u=0\}}(r\xi) \, dr \geq \int_{1/2}^1 r^{n-1} \chi_{\{u=0\}}(r\xi) \, dr,
\]

hence we find

\[
\left( \int_{\partial B_1} u \right)^2 \int_{1/2}^1 r^{n-1} \chi_{\{u=0\}}(r\xi) \, dr \leq C_n \int_{1/2}^1 |\nabla (u - v)(r\xi)|^2 \, dr \leq 2^{n-1} C_n \int_{1/2}^1 r^{n-1} |\nabla (u - v)(r\xi)|^2 \, dr.
\]

Integrating on \( \partial B_1 \), and recalling that \( u > 0 \) in \( B_{1/2} \), we obtain our claim. \( \square \)

**Theorem 4.2** Under the same hypotheses of Theorem 4.1, we consider a solution \( u \) to (M) and suppose that

(i) there exists an index \( i \) and a unit vector \( \nu \in \mathbb{R}^d \) such that \( \langle u(x) - l_i, \nu \rangle > 0 \) for all \( x \in \Omega \setminus L_i(u) \) (i.e., \( l_i \) is an extremal point of the convex hull of the image of \( u \));

(ii) \( \langle \nabla F(l_i), \nu \rangle < 0 \).

Then, \( u \) is locally Lipschitz on a neighbourhood of the level set \( L_i(u) \).

**Proof.** Choose a compact set \( K \) inside \( \Omega \), and let \( A \) be the open set where \( u \neq l_i \). Inside \( A \), \( u \) solves the Euler system

\[
2\triangle u = \beta \nabla F(u).
\]  

(28)

In particular, \( \triangle u \in L^\infty \) in \( A \) and hence \( u \) is \( W^{2,r}_0(A) \) for every \( r < +\infty \), therefore it is locally Lipschitz inside \( A \). Clearly, we have \( Du = 0 \) a.e. on each level set \( L_i(u) \), hence we only need to bound \( |Du| \) at those points of \( K \cap A \) which are close to \( \Omega \cap \partial A \).

As usual, assume \( l_i = 0 \) for more simplicity, and choose \( \nu \) as in (i). Consider a small ball with center at \( x_0 \in K \), tangent to the level set \( \{u = 0\} \), then increase its radius by a small quantity and denote by \( B_r(x_0) \) the resulting ball. For a more convenient notation, let us also suppose that \( x_0 = 0 \). Let \( v \) be the vector function which solves

\[
2\triangle v = \beta \nabla F(v)
\]

in \( B_r \) and which is equal to \( u \) along \( \partial B_r \). By reasoning as in Theorem 4.1, one immediately sees that \( |v| \) is uniformly bounded by a constant that does not depend on \( r \), hence by the regularity of \( F \) we obtain \( |\triangle v| \leq C \) on \( B_r \), where, of course, \( C > 0 \) does not depend on \( r \). Now, consider any component \( v^a \) of the vector function \( v \) and define \( v^+ \) and \( v^- \) as

\[
v^\pm(x) = v^a(x) \pm C(v^2 - |x - x_0|^2).
\]

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Since $|\Delta v^i| \leq C$, the two functions defined above are, respectively, superharmonic and subharmonic with the same boundary values as $v^i$. By taking $z$ as the harmonic function on $B_r$ coinciding with $v^i$ at the boundary, one obtains

$$v_i^- \leq z \leq v_i^+ \quad \text{on } B_r,$$

hence, by the maximum principle and the fact that $z = u^i$ on $\partial B_r$,

$$v_i^- \leq \sup_{B_r} z \leq \|u^i\|_\infty, \quad -\|u^i\|_\infty \leq \inf_{B_r} z \leq v_i^+,$$

where $\|u^i\|_\infty$ denotes the supremum of $|u^i|$ on $B_r$. From these last inequalities, one immediately sees that

$$\|v_i\|_\infty \leq \|u^i\|_\infty + Cr^2$$

for all $i = 1, \ldots, d$, and therefore, up to a multiplicative constant,

$$\|v\|_\infty \leq \|u\|_\infty + Cr^2. \quad (29)$$

Consider now the scalar function $w = (v, v)$ solving $2\Delta w = \beta (\nabla F(v), v)$ on $B_r$: for $r$ sufficiently small we have $\Delta w \leq 0$ on $B_r$, thanks to (ii), (29) and the fact that $\|u\|_\infty$ is small for small $r$ ($u$ is continuous and vanishes somewhere in $B_r$). Letting $u_r = (v, v)$, we obtain by Lemma 4.1

$$\left(\frac{1}{r} \int_{\partial B_r} u_r\right)^2 \{u_r = 0\} \cap B_r \leq C \int_{B_r} |\nabla (u_r - w)|^2. \quad (30)$$

On the other hand we have

$$\int_{B_r} |D(v - u)|^2 - \int_{B_r} |Du|^2 + \int_{B_r} |Dv|^2 = 2 \int_{B_r} \sum_{k=1}^d (\nabla (v^k - u^k), \nabla v^k)$$

$$= -2 \int_{B_r} \Delta v \cdot (v - u)$$

$$= \beta \int_{B_r} \nabla F(v) \cdot (u - v).$$

By the convexity of $F$, we have $(\nabla F(v), (u - v)) \leq F(u) - F(v)$, hence from the minimality of $u$ we get

$$\int_{B_r} |D(v - u)|^2 \leq \mathcal{E}(u, B_r) - \mathcal{E}(v, B_r) \leq \lambda |\{u = 0\} \cap B_r|. \quad (31)$$

Observing that $|D(u_r - w)|^2 \leq |D(u-v)|^2$ and that $|\{u_r = 0\} \cap B_r| \geq |\{u = 0\} \cap B_r|$, we find from (30) and (31)

$$\frac{1}{r} \int_{\partial B_r} u_r \leq C \lambda$$

and, by continuity of the trace, this inequality holds true also when $B_r$ is tangent to the level set $\{u = 0\}$. In this case, $B_r \subset A$ and hence $u$ solves (28) in $B_r$, therefore $u_r$ solves

$$2\Delta u_r = \beta \langle \nabla F(u), v \rangle \quad \text{in } B_r,$$

hence $|\Delta u_r| \leq C$ in $B_r$. Splitting $u_r = h + z$, with $h$ harmonic in $B_r$ and $z = 0$ along $\partial B_r$, we have the well-known estimates (note that $h \geq 0$)

$$|\nabla h(0)| \leq \frac{C}{r} \int_{\partial B_r} h = \frac{C}{r} \int_{\partial B_r} u_r, \quad |\nabla z(0)| \leq r \max_{B_r} |\Delta z| = r \max_{B_r} |\Delta u_r| \leq Cr,$$
and hence

$$|\nabla u_\nu(0)| \leq Cr + \frac{C}{r} \int_{\partial B_r} u_\nu \leq Cr + C\lambda.$$ 

By repeating the same argument for $d$ linearly-independent unit vectors $\nu_1, \ldots, \nu_d$ verifying (i) and (ii), we can recover $u$ as $N\tilde{u}$, where $\tilde{u} = (u_{\nu_1}, \ldots, u_{\nu_d})$ and $N$ is the $(d \times d)$-matrix whose inverse is

$$N^{-1} = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_d \end{bmatrix}$$

(Indeed, $\tilde{u} = N^{-1}u$, that is, $u = N\tilde{u}$). Then, by linearity, we obtain

$$Du = D(N\tilde{u}) = N \cdot D\tilde{u},$$

and finally

$$|Du| \leq |N| |D\tilde{u}| \leq C,$$

where $C > 0$ depends only on the data of the problem and on the image of $u$. Then, it follows that, for any compact set $K$ inside $\Omega$, $u$ is Lipschitz-continuous on a neighbourhood of $L_i(u) \cap K$, that is, it is locally Lipschitz near $L_i(u)$, as wanted. \hfill \Box

An application of the previous theorem is contained in the following Corollary, where $F(u)$ takes the special form $f(|u - L|)$ for a certain function $f$ and $L \in \mathbb{R}^d$:

**Corollary 4.2** Let $L$ be a given vector in $\mathbb{R}^d \setminus \{l_1, \ldots, l_m\}$ and suppose that $l_1, \ldots, l_m$ are extremal points of the convex hull of $\{L, l_1, \ldots, l_m\}$. If we consider a function $f : [0, +\infty) \to \mathbb{R}$ of class $C^1$, convex, strictly increasing and such that $f'(0) = 0$, then problem (M) corresponding to the energy

$$\mathcal{E}(u, \Omega) = \int_\Omega |Du|^2 + \beta \int_\Omega f(|u - L|)$$

admits solutions for all $\beta \geq 0$. Moreover, any solution is locally Lipschitz on $\Omega$.

**Proof.** Existence follows easily from Theorem 4.1 by setting $F(u) = f(|u - L|)$. Let $u$ be a solution to (M), then one can observe that the image of $u$ is contained in the convex hull of $L$ and the $l_i$’s, then conditions (i) and (ii) of Theorem 4.2 follow by extremality and by the fact that $\nabla F(l_i)$ is a (positive) multiple of the vector $l_i - L$. By Theorem 4.2, on any relatively compact subset $\omega$ of $\Omega$, $u$ is Lipschitz continuous near $L_i(u)$ for all $i$, while in the rest of $\omega$ one can estimate the modulus of the gradient as in the first lines of the proof of Theorem 4.2. \hfill \Box

**References**


